1Y ep0nzMR 0n {

hBu0yPS

2004^t7g 13e

\0kc2006/12/30

https://m-katsurada.sakura.ne.jp/lecture/kiso3/sekibun.pdf

v k!

1 0c0X0 0k **2**

2 N[zMR 0nQl_ **2**

2.1 zMR 0n}_b‘’ 2

2.2 nc zMR 3

2.3 ŁR zMR 3

2.4 Q" 3

2.5 N ¢ep0hc ep¢ep 3

2.6 S f } ¢ep 4

2.7 [epzMR 4

2.8 $1/\sqrt{a^2-x^2}$ 4

2.9 $1/(x^2+a^2)$ 4

2.10 $\sqrt{a^2-x^2}$ 5

2.10.1 V_bv 0k 0H0 (0J0Y0Y0) 5

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2.10.3 ŁR zMR 0k0 0e„l (S´ 0~0g) 6

2.11 $1/\sqrt{x^2+k}$ 6

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2.11.2 N ¢ep0k0 0 nc zMR 0u(0D0 (S´ —0‘0_0W[zMR T 0M) 7

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2.11.4 0S0nQl_0h0i0FN 0MT 0F0K 8

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2.12.1 ŁR zMR 0g$1/\sqrt{x^2+k}$ 0nzMR 0k^0w@(0J0Y0Y0) 9

2.12.2 S f } ¢ep0g nc zMR (S´) 9

2.12.3 N ¢ep0g nc zMR (S´) 10

2.12.4 nc $t-x=\sqrt{x^2+k}$ (S´) 11

2.12.5 _u(— e>ri} 0n_’w 12

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3.2 Y epY c (nc zMR) 12

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3.2.2 O 13
3.3 ÆR zMR 14

4 g t ¢ep0nzMR 16
4.1 g t ¢ep0nN [zMR 0cR {I ¢ep0g h0 0[0 — Leibniz 0n[t 16
4.2 [t 4.1 0n <f 17
4.3 ÆR R ep0x0nR ª 18
4.3.1 e„ 18
4.3.2 R k˝0nV epR ª 19
4.3.3 N 0D0k} 0jR k˝0 c 0cR ep_ 0x0nR ª 20
4.3.4 Y _ 0n $(x-a)$ 0n0 0 ›0x0n\U 21
4.3.5 Y _ 0n 2 k!_ 0n0 0 0 ›0 0x0n\U 21
4.3.6 0~0h0 22
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4.4.1 $R(e^x)$ 0nzMR 22
4.4.2 $R(x, \sqrt{ax+b})$ 0nzMR 22
4.4.3 N ¢ep0ng t ¢ep 22
4.4.4 2 k!q!t ¢ep 23
4.5 QwOSO 24
4.6 OY ˙ — [0nzMR { 26

5 N [zMR 0L0 lB0~0 0j0D0 X4T 26
5.1 0 lB0~0 0j0D0 0na Ts 26
5.2 iUQ zMR 27
5.3 zMR [ep¸ zMR kc_& 28
5.4 kc R ^ 0n[˜^ƒ ¢ep0nN [zMR 28

A S f } ¢ep0n'' Œ 29

B S f } ¢ep 29

1 0c0X0 0k

0S0ne fł0o1 Y ep ¢ep0nN [zMR 0n { 0k ¢0Y0 0Æ0 0g0B0 0 { } 0k ¢0W0f0os fBp„(2003 ^t) 0g0o[1]0 {I[fh!0nep[fIII 0neŸy fł0B0 0D0cUO L ˜0L [N g 0 R 0 0 0 0 S´ fł0g0B0 0 Y'[fN ^tk!0n_fizMR 0~0g0n{˜V 0g0 h!ep[f0nUO L ˜0k 0c0f0D0j0D0S0h0c0B0~0 0j0D(N ¢epz ^ƒ)0

2 N [zMR 0nQl_

2.1 zMR 0n} _b''

$$\int (f(x)+g(x))dx = \int f(x)\,dx + \int g(x)\,dx.$$

[1] 0 0F0Y0P h!ep[f0n0«0Œ0 ›0 0Ø0 0LY 0 0n0g0 0S0S0k 0y0f0D0 0S0h0db z¸0W0j0C0j0 S ''0L 0D0

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

2.2 nc zMR

$$\int f(g(x))g'(x)\,dx = \int f(t)\,dt, \quad t = g(x).$$

2.3 ŁR zMR

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

2.4 Q"

$$\int x^{\alpha}\,dx = \begin{cases} \dfrac{x^{\alpha+1}}{\alpha+1} + C & (\alpha \neq -1) \\ \log|x| + C & (\alpha = -1) \end{cases}$$

2.5 N ¢ep0hc ep ¢ep

$$\int \cos x\,dx = \sin x + C.$$

$$\int \sin x\,dx = -\cos x + C.$$

$$\int \frac{dx}{\cos^2 x} = \tan x + C.$$

$$\int \frac{dx}{\sin^2 x} = -\cot x + C.$$

$$\int e^x\,dx = e^x + C.$$

$a > 0$, $a \neq 1$ 0j0 a 0k[0W0f0

$$\int a^x\,dx = \frac{a^x}{\log a} + C.$$

0 0c0h0 0S0 0 0H0 0 0 0c0 $a^x = (e^{\log a})^x = e^{x\log a}$ 0h0D0Ff ł0Mc 0H0kac0 0 e„0L0 0D0

2.6 S f } ¢ep

S f } ¢ep0oe^x 0ng t _ 0j0n0g0 _ 0g 0y0 0 0F0k_¯0ZN [zMR 0IQœge0 0L0 N ¢ep0k ^O<0W0_}Pg 0L_ 0 0 0 0

$$\int \cosh x\,dx = \sinh x + C,$$

$$\int \sinh x\,dx = \cosh x + C,$$

$$\int \frac{dx}{\cosh^2 x} = \tanh x + C,$$

$$\int \frac{dx}{\sinh^2 x} = \coth x + C.$$

2.7 [epzMR

$$\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C.$$

ry0k

$$\int \tan x\,dx = -\log|\cos x| + C,$$

$$\int \cot x\,dx = \log|\sin x| + C.$$

2.8 $1/\sqrt{a^2-x^2}$

Y'[f0k0j0c0f0K0 0c0_e 0W0DQl_ 0g0B0 ([zMR 0n { 0o h!ep[f0g0 Qœ0fge0_)0

$$\int \frac{dx}{\sqrt{1-x^2}} = \operatorname{Arcsin} x + C.$$

$a > 0$ 0k[0W0f

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \operatorname{Arcsin}\left(\frac{x}{a}\right) + C.$$

N8f 0 0 0 $x = a\sin\theta$ $(-\pi/2 < x < \pi/2)$ 0h0D0FY epY c 0 0H0_e„0L0 0D0

2.9 $1/(x^2+a^2)$

0S0 0 Y'[f0k0j0c0f0K0 0c0_e 0W0DQl_ 0g0B0 ([zMR 0n { 0o &T N)0

$$\int \frac{dx}{x^2+1} = \operatorname{Arctan} x + C.$$

$a > 0$ 0k[0W0f

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{Arctan}\left(\frac{x}{a}\right) + C.$$

0S0 0 N8f 0 0 0 $x = a\tan\theta$ $(-\pi/2 < x < \pi/2)$ 0h0D0FY epY c 0 0H0_e„0L0 0D0

2.10 $\sqrt{a^2-x^2}$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(a^2 \operatorname{Arcsin}\left(\frac{x}{a}\right) + x\sqrt{a^2-x^2}\right) + C.$$

N8f 0c0 0 Sq z0g0B0 0F0 r0 0j0 0 e„0L0B0 0 0i0 0 ˝ 0 ‘0L0 0He„0h0W0f0c0 V _b v 0ja Ts0 0d0Q0 2.10.1 0L0JR 0 0g0B0 (0Y0 0kQ 0L0 0 0nV _b0n bzM0j0n0g0 [zMR 0j0 0p h!ep[f0 0 0°0g0B0)0

2.10.1 V _bv 0k 0H0 (0J0Y0Y0)

$$\begin{aligned}\int_0^X \sqrt{a^2-x^2}\,dx &= \text{“}y = \sqrt{a^2-x^2}\ 0\mathrm{h}\,x \quad \text{Ł}, x=0, x=X\ 0\mathrm{gV}\ 0\sim 0\ 0 \quad \text{ŁR}\ 0\mathrm{n}\ \mathrm{bzM}\text{”}\\ &= \frac{1}{2}a^2\theta + \frac{1}{2}X\sqrt{a^2-X^2} \quad (\theta\ 0\mathrm{cbG_b0nN}-_\hat{}\ )\\ &= \frac{1}{2}\left(a^2 \operatorname{Arcsin}\left(\frac{X}{a}\right) + X\sqrt{a^2-X^2}\right).\end{aligned}$$

2.10.2 nc $x = a\sin\theta$ 0k0 0 e„l (_ u(0LR„00)

$x = a\sin\theta\ (\theta \in [-\pi/2, \pi/2])$ 0h0J0C0h0

$$dx = a\cos\theta\,d\theta, \quad \sqrt{a^2-x^2} = a\cos\theta$$

0g0B0 0K0 0

$$\begin{aligned}\int \sqrt{a^2-x^2}\,dx &= \int a\cos\theta\cdot a\cos\theta\,d\theta = a^2\int \cos^2\theta\,d\theta = a^2\int \frac{1+\cos 2\theta}{2}\,d\theta\\ &= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{a^2}{2}\left(\theta + \sin\theta\cos\theta\right) + C\\ &= \frac{a^2}{2}\left(\operatorname{Arcsin}\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2}\right) + C\\ &= \frac{1}{2}\left(a^2 \operatorname{Arcsin}\left(\frac{x}{a}\right) + x\sqrt{a^2-x^2}\right) + C.\end{aligned}$$

2.10.3 ŁR zMR 0k0 0 e„l (S ´ 0~0g)

$$\begin{aligned}\int \sqrt{a^2-x^2}\,dx &= \int (x)'\cdot\sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} - \int x\left(\sqrt{a^2-x^2}\right)'dx\\ &= x\sqrt{a^2-x^2} - \int x\frac{-x}{\sqrt{a^2-x^2}}\,dx\\ &= x\sqrt{a^2-x^2} + \int \frac{x^2-a^2+a^2}{\sqrt{a^2-x^2}}\,dx\\ &= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\,dx + a^2\int \frac{dx}{\sqrt{a^2-x^2}}\end{aligned}$$

0 0

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}}\right) = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2 \operatorname{Arcsin}\left(\frac{x}{a}\right)\right) + C.$$

2.11 $1/\sqrt{x^2+k}$

$k \neq 0$ 0g0B0 0h0M0

$$\int \frac{dx}{\sqrt{x^2+k}} = \log\left|x+\sqrt{x^2+k}\right| + C.$$

($k > 0$ 0n0h0M0o}v[P$0n S 0cN 0g0B0 0 0~0_ $k < 0$ 0g0B0c0f0 $x \geq \sqrt{|k|}$ 0n0h0M0c0 0 0o 0 }v[P$0n S 0cN 0g0B0 0 $k < 0$ 0g0K0d $x \leq -\sqrt{|k|}$ 0n0h0M0o}v[P$0n S 0cw 0Q0j0D0)

0S0nQl_ 0k0c0B0~0 "0W0 0n0j0DN"0LY 0D0g0B0 0F0

0 0a0 0 0 S "0 _fiR 0W0_0] "0n «zMR ¢ep0k0j0 0S0h0 x" 0Y0 0p <f 0g0M0 0 0Q0' 0L0 S "0L0]0 0{0i 0H0 0Y0D_b0g0c0j0D0n0g0 OU0K c0DbK0c0j0D0K0h 0H0 0n0o Œq60g0B 0 0F0

{F 0cN _ h!0nep[f0nc im0g $(t - x = \sqrt{x^2+k}$ 0h n0O\ Qœl 0) 0c0_0L0 0H0e0 0O 0f S^{a}0W0_ 0H0L0B0 0 k 0n{&S 0k_ 0X0f0 $k = a^2$ 0nX4T 0k0o $x = a\tan t$, $k = -a^2$ 0nX4T 0k 0o $x = a/\sin t$ 0h0D0F nc 0 0B0 0 0S0 0c0K0j0 N ,''0L0B0 0 0n0n0 0]0 0{0i|!SX0k0c0j0 0Z0 lz[bS0h0o 0D0e0 0D0

Mathematica 0j0i0 O 0F0h0 $k > 0$ 0nX4T 0k}Pg 0L S f } ¢ep $\sinh^{-1} =$ ArcSinh[] 0g h y:0U0 0 0 x"0K0kS f } ¢ep0 u(0D0f nc 0Y0 0h0D0F0 0 e„0c0B0 0L0 0]0 0{0i|!SX0h0o 0H 0j0D0h' 0F0

k 0 0 0g0j0D[[ep0 a 0 kc[ep0h0W0f

$$I = \int \frac{dx}{\sqrt{x^2+k}}, \quad I_p = \int \frac{dx}{\sqrt{x^2+a^2}}, \quad I_m = \int \frac{dx}{\sqrt{x^2-a^2}}$$

0h0J0C0

I_p, I_m 0o_ 0Y0 0 0F0k S f } ¢ep0 u(0D0f}Pg 0 h0Y0S0h0 0g0M0 (ry0kR 0 0j0D)0

2.11.1 S f } ¢ep0k0 0 nc zMR 0 u(0D0

(S f } ¢ep0k0d0D0f0c0 B 0 S´qg0[0 0)

I_p 0nX4T $x = a\sinh t$ 0h0J0C0h0

$$dx = a\cosh t\,dt, \quad \sqrt{x^2+a^2} = \sqrt{a^2(\sinh^2 t+1)} = \sqrt{a^2\cosh^2 t} = a\cosh t$$

0g0B0 0K0 0

$$I_p = \int \frac{a\cosh t}{a\cosh t}dt = \int dt = t + C = \sinh^{-1}\left(\frac{x}{a}\right) + C = \log\left|\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2+1}\right| + C$$
$$= \log\left|x+\sqrt{x^2+a^2}\right| - \log a + C = \log\left|x+\sqrt{x^2+a^2}\right| + C'.$$

0S0n I_p 0k0d0D0f0c0 S f } ¢ep0 O 0c0_

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$$

0n_b0g 0H0f0J0D0f0 c0D0K0 0W0 0j0D0

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

0hN&0y0 0h bv}0D0

I_m 0nX4T $x^2 - a^2 \geq 0$ 0g0B0 0_0 0k0o0 $x \geq a$ 0~0_0o $x \leq -a$ 0g0B0 _¯ 0L0B0 0 $x \geq a$ 0n0h0M0 $x = a\cosh t$ $(t \geq 0)$ 0h0J0C0h0 $(a\sinh t \geq 0$ 0kl‡a 0W0f)

$$dx = a\sinh t\,dt, \quad \sqrt{x^2 - a^2} = \sqrt{a^2(\cosh^2 t - 1)} = \sqrt{a^2\sinh^2 t} = a\sinh t$$

0g0B0 0K0 0

$$\begin{aligned} I_m &= \int \frac{a\sinh t}{a\sinh t}dt = \int dt = t + C = \cosh^{-1}\left(\frac{x}{a}\right) + C \\ &= \log\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1}\right) + C \\ &= \log\left(x + \sqrt{x^2 - a^2}\right) - \log a + C = \log\left(x + \sqrt{x^2 - a^2}\right) + C'. \end{aligned}$$

$x \leq -a$ 0n0h0M0o $x = -a\cosh t$ $(t \geq 0)$ 0h0J0D0f0

$$dx = -a\sinh t, \quad \sqrt{x^2 - a^2} = a\sinh t$$

0g0B0 0K0

$$\begin{aligned} I_m &= \int \frac{-a\sinh t}{a\sinh t}dt = -t + C = -\cosh^{-1}\left(\frac{-x}{a}\right) + C \\ &= -\log\left(\frac{-x}{a} + \sqrt{\left(\frac{-x}{a}\right)^2 - 1}\right) + C \\ &= -\log\left(-x + \sqrt{x^2 - a^2}\right) + \log a + C \\ &= \log\left(-x + \sqrt{x^2 - a^2}\right)^{-1} + \log a + C \\ &= \log\frac{-\left(x + \sqrt{x^2 - a^2}\right)}{a^2} + \log a + C \\ &= \log\left[-\left(x + \sqrt{x^2 - a^2}\right)\right] - \log a + C \\ &= \log\left|x + \sqrt{x^2 - a^2}\right| - \log a + C. \end{aligned}$$

}Pg 0 $\cosh^{-1}$ 0gf‡0M0_0Q0 0p

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{sign} x \cdot \cosh^{-1}\left(\frac{|x|}{a}\right) = \begin{cases} \cosh^{-1}\left(\dfrac{x}{a}\right) & (x > a) \\ -\cosh^{-1}\left(\dfrac{-x}{a}\right) & (x < -a). \end{cases}$$

2.11.2 N ¢ep0k0 0 nc zMR 0 u(0D0 (S´ — 0‘0_0W[zMR T 0M)

I_p 0nX4T $x = a\tan t$ 0h0J0C0h0

$$dx = \frac{a}{\cos^2 t}dt, \quad \sqrt{x^2 + a^2} = \sqrt{a^2(\tan^2 t + 1)} = \frac{a}{\cos t}$$

0g0B0 0K0

$$I_p = \int \frac{\cos t}{a}\frac{a}{\cos^2 t}\,dt = \int \frac{dt}{\cos t} = \int \frac{\cos t}{1 - \sin^2 t}\,dt.$$

$u = \sin t$ 0h0J0C0h0

$$I_p = \int \frac{du}{1-u^2} = \frac{1}{2}\int \left(\frac{1}{1+u} + \frac{1}{1-u}\right) du = \frac{1}{2}\log\frac{1+u}{1-u} + C$$
$$= \frac{1}{2}\log\frac{(1+\sin t)^2}{\cos^2 t} + C = \log\left|\frac{1+\sin t}{\cos t}\right| + C = \log\left|\tan t + \sqrt{\tan^2 t + 1}\right| + C$$
$$= \log\left|\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 + 1}\right| + C = \log\left|x + \sqrt{x^2+a^2}\right| - \log a + C = \log\left|x + \sqrt{x^2+a^2}\right| + C'.$$

I_m 0nX4T $x = \dfrac{a}{\sin t}$ 0h0J0C0h0

$$dx = \frac{-a\cos t}{\sin^2 t}dt, \quad \sqrt{x^2-a^2} = \sqrt{a^2\left(\frac{1}{\sin^2 t} - 1\right)} = a\frac{\cos t}{\sin t}$$

0g0B0 0K0

$$I_m = \int \frac{\sin t}{a\cos t}\cdot\frac{-a\cos t}{\sin^2 t}dt = -\int \frac{dt}{\sin t} = \int \frac{-\sin t}{1-\cos^2 t}\,dt.$$

$u = \cos t$ 0h0J0C0h0

$$I_p = \int \frac{du}{1-u^2} = \frac{1}{2}\log\frac{1+u}{1-u} + C$$
$$= \frac{1}{2}\log\frac{(1+\cos t)^2}{\sin^2 t} + C = \log\left|\frac{1+\cos t}{\sin t}\right| + C = \log\left|\cot t + \frac{1}{\sin t}\right| + C$$
$$= \log\left|\frac{1}{a}\sqrt{x^2-a^2} + \frac{x}{a}\right| + C = \log\left|x + \sqrt{x^2-a^2}\right| - \log a + C = \log\left|x + \sqrt{x^2-a^2}\right| + C'.$$

2.11.3 nc $t - x = \sqrt{x^2+k}$ (0Y0Y0 0 0 0)

({I[fh!0nc im0g 0c0_— 0 0C0S0 0j0n0 eY0H0f0C0 0_0 0n0‘0h‘ 0F0)

$$t - x = \sqrt{x^2+k}$$

0h0J0C0h0

$$\frac{dt}{dx} = 1 + \frac{x}{\sqrt{x^2+k}} = \frac{x+\sqrt{x^2+k}}{\sqrt{x^2+k}} = \frac{t}{\sqrt{x^2+k}}$$

0g0B0 0K0

$$\frac{dt}{t} = \frac{dx}{\sqrt{x^2+k}}.$$

0 0H0k

$$I = \int \frac{dt}{t} = \log t + C = \log\left|x + \sqrt{x^2+k}\right| + C.$$

2.11.4 0S0nQl_ 0h0i0FN 0MT 0F0K

r0 ‘'0 0‘}Pg 0 y`0n}P 0c0 0H0 0 0_0‘0WSq0j0D0n0gO 0FRM0ki { 0Y0 00 0h0D0F0 0n 0g0B0 0 0d0~0

$$\int \frac{dx}{\sqrt{x^2+k}} = \log\left|x + \sqrt{x^2+k}\,\right|$$

0 0H0 O 0F 0cS "0 _fiR 0W0f] "0k0j0 0S0h0 0`0S0^0fl0Y0 0 0h0D0F0S0h0g0B0 0

2.12 $\sqrt{x^2+k}$

$k \neq 0$ 0n0h0M0

$$\int \sqrt{x^2+k}\,dx = \frac{1}{2}\left(x\sqrt{x^2+k} + k\log\left|x+\sqrt{x^2+k}\right|\right).$$

0S0 0~0_ bP 0g0B0 0 $1/\sqrt{x^2+k}$ 0nzMR 0hT i 0 r0 0j0 0 e„0L0B0 0L0 | !SX0je„l 0c0j0D (\ 0j0C0h0 w 0 0j0D)0

f [{V0h0W0f0c[2]0 ŁR zMR 0 u(0D0f $1/\sqrt{x^2+k}$ 0k^0w@0U0 [0 $1/\sqrt{x^2+k}$ 0nzMR 0o 0H0 0 0h0D0F0n0L c0U0]0F0g0B0 0

k 0 0 0g0j0D[[ep0 a 0 kc[ep0h0W0f

$$I = \int \sqrt{x^2+k}\,dx, \quad I_p = \int \sqrt{x^2+a^2}\,dx, \quad I_m = \int \sqrt{x^2-a^2}\,dx$$

0h0J0C0

2.12.1 ŁR zMR 0g $1/\sqrt{x^2+k}$ 0nzMR 0k^0w@ (0J0Y0Y0)

$$\begin{aligned} I = \int \sqrt{x^2+k}\,dx &= \int (x)'\cdot\sqrt{x^2+k}\,dx = x\sqrt{x^2+k} - \int x\cdot\left(\sqrt{x^2+k}\right)'dx \\ &= x\sqrt{x^2+k} - \int x\frac{x}{\sqrt{x^2+k}}\,dx = x\sqrt{x^2+k} - \int \frac{x^2+k-k}{\sqrt{x^2+k}}\,dx \\ &= x\sqrt{x^2+k} - \int \sqrt{x^2+k}\,dx + k\int \frac{dx}{\sqrt{x^2+k}} \\ &= x\sqrt{x^2+k} - I + k\int \frac{dx}{\sqrt{x^2+k}} \end{aligned}$$

0 0

$$I = \frac{1}{2}\left(x\sqrt{x^2+k} + k\int \frac{dx}{\sqrt{x^2+k}}\right) = \frac{1}{2}\left(x\sqrt{x^2+k} + k\log\left|x+\sqrt{x^2+k}\right|\right) + C.$$

2.12.2 S f } ¢ep0g nc zMR (S´)

I_p 0nX4T $x = a\sinh t$ 0h0J0C0h (e 0k 0_0 0F0k)

$$dx = a\cosh t\,dt, \quad \sqrt{x^2+a^2} = a\cosh t$$

[2] a¶ S0 0_0D0j0 0n0c0 ep[f0nc im0geY0H0 0 0n0g0c0j0D0h0D0FN”0 0D0 0g0B0 0F0L0 Y’0M0jw X0nOS|ß0h 0i0F0d0M0B0F0K0h0D0F0S0h0 Y’N 0g0B0 0 [v 0k0c0 0≠0 0 0 0 0¿0 0h0JS¸0‘0a0k0j0 0 0S0h0g[Q 0Y0 0n0L c0D0h‘ 0F0L0 ŒR 0n «OS0‘0Q0gOU0h0K0Y0 0h0D0F0n0j0 0p0 0i0S0~0g_ 0 0H0f0 0i0S0K0 bK0g\ Qœ0Y0 0KN RM0k 0H0f0J0CaK^f0o ˝ 0g0B0 0

0g0B0 0K0 0

$$
\begin{aligned}
I_p &= a^2 \int \cosh^2 dt = a^2 \int \frac{\cosh 2t + 1}{2} dt \\
&= \frac{a^2}{2}\left(t + \frac{1}{2}\sinh 2t\right) + C = \frac{a^2}{2}\left(t + \sinh t \cosh t\right) + C = \frac{a^2}{2}\left(t + \sinh t \sqrt{\sinh^2 t + 1}\right) + C \\
&= \frac{a^2}{2}\left(\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{\left(\frac{x}{a}\right)^2 + 1}\right) + C \\
&= \frac{1}{2}\left(a^2 \log\left|\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 + 1}\right| + x\sqrt{x^2 + a^2}\right) + C \\
&= \frac{1}{2}\left(x\sqrt{x^2 + a^2} + a^2 \log\left|x + \sqrt{x^2 + a^2}\right|\right) - \frac{a^2}{2}\log a + C \\
&= \frac{1}{2}\left(x\sqrt{x^2 + a^2} + a^2 \log\left|x + \sqrt{x^2 + a^2}\right|\right) + C'.
\end{aligned}
$$

I_m 0nX4T $x = a\cosh t$ 0h0J0C0h(e 0k 0_0 0F0k)

$$dx = a\sinh t\, dt, \quad \sqrt{x^2 - a^2} = a\sinh t$$

0g0B0 0K0 0

$$
\begin{aligned}
I_m &= a^2 \int \sinh^2 t\, dt = a^2 \int \frac{\cosh 2t - 1}{2}\, dt \\
&= \frac{a^2}{2}\left(\frac{1}{2}\sinh 2t - t\right) + C = \frac{a^2}{2}\left(\sinh t \cosh t - t\right) + C = \frac{a^2}{2}\left(\cosh t\sqrt{\cosh^2 t - 1} - t\right) + C \\
&= \frac{a^2}{2}\left(\frac{x}{a}\sqrt{\left(\frac{x}{a}\right)^2 - 1} - \cosh^{-1}\left(\frac{x}{a}\right)\right) + C \\
&= \frac{1}{2}\left(x\sqrt{x^2 - a^2} - a^2 \log\left|\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1}\right|\right) + C \\
&= \frac{1}{2}\left(x\sqrt{x^2 - a^2} - a^2 \log\left|x + \sqrt{x^2 - a^2}\right|\right) + \frac{a^2}{2}\log a + C \\
&= \frac{1}{2}\left(x\sqrt{x^2 - a^2} - a^2 \log\left|x + \sqrt{x^2 - a^2}\right|\right) + C'.
\end{aligned}
$$

2.12.3 N ¢ep0g nc zMR (S´)

I_p 0nX4T $x = a\tan t$ 0h0J0C0h0

$$dx = \frac{a}{\cos^2 t}\, dt, \quad \sqrt{x^2 + a^2} = \sqrt{a^2(\tan^2 t + 1)} = \frac{a}{\cos t}$$

0g0B0 0K0 0

$$I_p = \int \frac{a}{\cos t} \cdot \frac{a}{\cos^2 t}\, dt = a^2 \int \frac{dt}{\cos^3 t} = a^2 \int \frac{\cos t}{(1 - \sin^2 t)^2} dt.$$

$u = \sin t$ 0h0J0C0h0

$$
\begin{aligned}
I_p &= a^2 \int \frac{du}{(1 - u^2)^2} = \frac{a^2}{4}\int \left(\frac{1}{(u-1)^2} + \frac{1}{u+1} - \frac{1}{u-1} + \frac{1}{(u+1)^2}\right) du \\
&= \frac{a^2}{4}\left(\frac{-1}{u-1} + \log\left|\frac{1+u}{1-u}\right| + \frac{-1}{u+1}\right) + C \\
&= \frac{a^2}{4}\left(\frac{2u}{1-u^2} + \log\frac{(1+u)^2}{1-u^2}\right) + C.
\end{aligned}
$$

0S0S0g

$$\frac{2u}{1-u^2} = \frac{2\sin t}{\cos^2 t} = 2\tan t\sqrt{\tan^2 t+1} = 2\frac{x}{a}\sqrt{\left(\frac{x}{a}\right)^2+1},$$
$$\log\frac{(1+u)^2}{1-u^2} = \log\frac{(1+\sin t)^2}{\cos^2 t} = 2\log\frac{1+\sin t}{\cos t} = 2\log\left|\frac{1}{\cos t}+\tan t\right|$$
$$= 2\log\left|\tan t+\sqrt{\tan^2 t+1}\right| = 2\log\left|\frac{x}{a}+\sqrt{\left(\frac{x}{a}\right)^2+1}\right|$$
$$= 2\left(\log\left|x+\sqrt{x^2+a^2}\right| - \log|a|\right)$$

0g0B0 0K0 0

$$I_p = \frac{1}{2}\left(x\sqrt{x^2+a^2} + a^2\log\left|x+\sqrt{x^2+a^2}\right|\right) + C'.$$

I_m 0nX4T $x = \dfrac{a}{\sin t}$ 0h0J0C0h

$$dx = -\frac{a\cos t}{\sin^2}\,dt, \quad \sqrt{x^2-a^2} = \sqrt{a^2\frac{1-\sin^2 t}{\sin^2 t}} = \frac{a\cos t}{\sin t}$$

0g0B0 0K0 0

$$I_m = \int\frac{a\cos t}{\sin t}\cdot\frac{-a\cos t}{\sin^2 t}\,dt = -a^2\int\frac{\cos^2 t}{\sin^3 t}\,dt = a^2\int\frac{\cos^2 t}{(1-\cos^2 t)^2}(-\sin t)dt.$$

$\cos t = u$ 0h0J0C0h0

$$\begin{aligned} I_m &= a^2\int\frac{u^2}{(1-u^2)^2}du \\ &= \frac{a^2}{4}\int\left(\frac{1}{(u+1)^2}+\frac{-1}{u+1}+\frac{1}{u-1}+\frac{1}{(u-1)^2}\right)du \\ &= \frac{a^2}{4}\left(\frac{-1}{u+1}-\log(u+1)+\log(u-1)+\frac{-1}{u-1}\right)+C \\ &= \frac{a^2}{4}\left(\frac{2u}{1-u^2}+\log\frac{1-u}{1+u}\right)+C. \end{aligned}$$

2.12.4 nc $t-x=\sqrt{x^2+k}$ (S´)

$t-x=\sqrt{x^2+k}$ 0h0J0C0h0 $t^2-2tx+x^2=x^2+k$ 0g0B0 0K0 0 x 0k0d0D0f ª0D0f0

$$x = \frac{t^2-k}{2t} = \frac{1}{2}\left(t-\frac{k}{t}\right).$$

0 0H0k

$$dx = \frac{1}{2}\left(1+\frac{k}{t^2}\right)dt = \frac{t^2+k}{2t^2}dt.$$

N e„0g0

$$\sqrt{x^2+k} = t-x = t-\frac{t^2-k}{2t} = \frac{t^2+k}{2t}$$

0h0j0 0K0 0

$$I = \int \frac{t^2+k}{2t} \cdot \frac{t^2+k}{2t^2} dt = \int \frac{(t^2+k)^2}{4t^3} dt = \frac{1}{4} \int \left(t + \frac{2k}{t} + \frac{k^2}{t^3} \right) dt$$
$$= \frac{1}{4} \left(\frac{t^2}{2} + 2k \log|t| - \frac{k^2}{2t^2} \right) + C$$
$$= \frac{1}{4} \left(\frac{t^4-k^2}{2t^2} + 2k \log|t| \right) + C = \frac{1}{2} \left(\frac{t^2+k}{2t} \cdot \frac{t^2-k}{2t} + k \log|t| \right) + C$$
$$= \frac{1}{2} \left(x\sqrt{x^2+k} + k \log\left| x + \sqrt{x^2+k} \right| \right) + C.$$

2.12.5 _ u(— e>ri} 0n_' w

$y = x^2$ 0n $x = a$ 0K0 $x = b$ 0~0g0n_' w

$$L = \int_a^b \sqrt{1+(y')^2}\, dx = \int_a^b \sqrt{1+4x^2}\, dx = 2\int_a^b \sqrt{x^2+\frac{1}{4}}\, dx$$
$$= 2\left[\frac{1}{2} \left(x\sqrt{x^2+\frac{1}{4}} + \frac{1}{4} \log\left| x + \sqrt{x^2+\frac{1}{4}} \right| \right) \right]_a^b = \left[x\sqrt{x^2+\frac{1}{4}} \right]_a^b + \frac{1}{4} \left[\log\left| x + \sqrt{x^2+\frac{1}{4}} \right| \right]_a^b.$$

3 } ^3

3.1 x^α 0nzMR

$f(x) = \sum_\alpha a_\alpha x^\alpha$ 0n_b0 0W0f0D0 0 0n0

$$\int \frac{1}{\sqrt[3]{x}}\, dx, \quad \int (x^{\frac{1}{2}} + 2x^{\frac{1}{3}} + 3x^{\frac{1}{4}}) dx, \quad \int \frac{1+2x^2+3x^3}{x^2}\, dx$$

0j0i0

3.2 Y epY c (nc zMR)

3.2.1 }ˇ

$$I = \int g(f(x)) f'(x)\, dx$$

0h0D0F_b0k0j0c0f0D0 0p $u = f(x)$ 0h0J0C0S0h0g0

$$I = \int g(u)\, du$$

0h0j0 0

(1) $f(x) = ax^2 + bx + c$ 0n0h0M $f'(x) = 2ax + b$ 0 0 $(2ax+b)\, dx = du.$ ry0k

$$\int g(x^2+c)x\, dx = \int g(u)\frac{u}{2}\, du, \quad u = x^2 + c.$$

(2) $f(x) = \sin x$ 0n0h0M $f'(x) = \cos x$ 0 0 $\cos x\,dx = du$.

$$\int g(\sin x)\cos x\,dx = \int g(u)\,du, \quad u = \sin x.$$

T i 0k

$$\int g(\cos x)\sin x\,dx = -\int g(u)\,du, \quad u = \cos x.$$

(3) $f(x) = \log x$ 0n0h0M $f'(x) = 1/x$ 0 0 $\dfrac{dx}{x} = du$.

$$\int \frac{g(\log x)}{x}\,dx = \int g(u)\,du, \quad u = \log x.$$

3.2.2 o

o 3.1

$$I = \int (2x+3)^5\,dx$$

$u = 2x+3$ 0h0J0C0h $du = 2\,dx$ 0 0 $dx = \dfrac{du}{2}$ 0 '0K0 0

$$I = \int u^5 \frac{du}{2} = \frac{1}{2}\cdot\frac{u^6}{6} + C = \frac{(2x+3)^6}{12} + C.$$

o 3.2

$$I = \int \frac{x}{(x^2+1)^2}\,dx.$$

$u = x^2+1$ 0h0J0C0h $du = 2x\,dx$ 0 '0K0 $x\,dx = \dfrac{du}{2}$ 0j0n0g0

$$I = \int \frac{1}{u^2}\cdot\frac{du}{2} + C = \frac{-u^{-1}}{2} + C = \frac{-1}{2(x^2+1)} + C.$$

o 3.3

$$I = \int x^2\sqrt{2x+1}\,dx.$$

$u = 2x+1$ 0h0J0C0h $du = 2dx$, $x = \dfrac{u-1}{2}$ 0g0B0 0K0

$$\begin{aligned} I &= \int \left(\frac{u-1}{2}\right)^2 \cdot u^{1/2}\frac{du}{2} = \frac{1}{8}\int (u-1)^2 u^{1/2}\,du \\ &= \frac{1}{8}\int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right) du \\ &= \frac{1}{8}\left(\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C = \frac{1}{28}u^{7/2} - \frac{1}{10}u^{5/2} + \frac{1}{12}u^{3/2} + C. \end{aligned}$$

0B0 0D0o $u = \sqrt{2x+1}$ 0h0J0C0h0 $x = (u^2-1)/2$ 0g $dx = u\,du$

$$I = \int \left(\frac{u^2-1}{2}\right)^2 u\cdot u\,du = \frac{1}{4}\int (u^6 - 2u^4 + u^2)du = \cdots$$

○ **3.4**

$$I = \int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx.$$

$u = \cos x$ 0h0J0C0h0 $du = -\sin x\,dx$ 0g0B0 0K0 0

$$I = \int (1-u^2)(-du) = \int (u^2-1)\,du = \frac{u^3}{3} - u + C = \frac{\cos^3 x}{3} - \cos x + C.$$

○ **3.5**

$$\int \frac{dx}{\cos x} = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{(\sin x)'}{1-\sin^2 x}\,dx = \int \frac{du}{1-u^2} = \frac{1}{2}\int \left(\frac{1}{1+u} + \frac{1}{1-u}\right)$$
$$= \frac{1}{2}\log\left|\frac{1+u}{1-u}\right| + C = \frac{1}{2}\log\frac{1+\sin x}{1-\sin x} + C.$$

○ **3.6** $u = \log x$ 0h0J0C0h $du = \frac{1}{x}\,dx$ 0 0

$$\int \frac{(\log x)^3}{x}\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{(\log x)^4}{4} + C.$$

T i 0k $\displaystyle\int \frac{\sin(\log x)}{x}\,dx$ 0j0i0 zMR 0g0M0 0

○ **3.7**

$$I = \int \frac{dx}{\sqrt{-x^2+2x+3}} = \int \frac{dx}{\sqrt{4-(x-1)^2}}$$

0h0j0 0n0g $x-1 = 2\sin\theta$ 0h0J0C0h

$$dx = 2\cos\theta\,d\theta, \quad \sqrt{4-(x-1)^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$$

0g0B0 0K0 0

$$I = \int d\theta = \theta + C = \operatorname{Arcsin}\left(\frac{x-1}{2}\right) + C.$$

3.3 ŁR zMR

“Y _ × c ep ¢ep0~0_0o sin, cos” 0h0D0F0¿0↗0 0c0 ŁR zMR 0 0Y0 0S0h0gY _ 0 m S» 0Y0 0S0h0L0g0M0 0

○ **3.8**

$$\int xe^x\,dx = \int (e^x)'x\,dx = e^x x - \int e^x\cdot 1\,dx = e^x x - e^x + C.$$

○ **3.9**

$$\int x\sin x\,dx = \int (-\cos x)'\cdot x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

○ **3.10**

$$\int x^2\cos x\,dx = \int x^2\cdot(\sin x)'\,dx = x^2\sin x - \int 2x\cdot\sin x\,|dx = x^2\sin x - 2\int x(-\cos x)'\,dx$$
$$= x^2\sin x + 2\left(x\cos x - \int 1\cdot\cos x\,dx\right) = x^2\sin x + 2x\cos x - 2\sin x + C.$$

Y _ × log x 0nX4T 0c0 log x 0ne„0 m 0Y0h0F0~0O L0C0

O **3.11**

$$\int x\log x\,dx=\int\left(\frac{x^2}{2}\right)'\log x\,dx=\frac{x^2}{2}\log x-\int\frac{x^2}{2}\frac{1}{x}\,dx=\frac{x^2}{2}\log x-\int\frac{x}{2}\,dx=\frac{x^2}{2}\log x-\frac{x^2}{4}+C.$$

0S0nryR%0jX4T 0L0 g T 0j log x 0nzMR 0g0B0 :

O **3.12**

$$\int\log x\,dx=\int(x)'\cdot\log x\,dx=x\log x-\int x\cdot\frac{1}{x}\,dx=x\log x-x+C.$$

T i 0k0W0f $\int$ Arctan $x\,dx$, $\int$ Arcsin $x\,dx$ 0j0i0L { 0g0M0 0

0U0f0 c ep ¢ep × sin x (0~0_0o cos x) 0cN V ÆR zMR 0Y0 0h ŒR Œ «0Ls 0 0 0f ª0Q0 0n 0g0B0c0_0

O **3.13**

$$\begin{aligned}I&=\int e^x\sin x\,dx=\int(e^x)'\sin x\,dx=e^x\sin x-\int e^x\cos x\,dx\\&=e^x\sin x-e^x\cos x-\int e^x\sin x\,dx\\&=e^x\sin x-e^x\cos x-I\end{aligned}$$

0 0

$$I=\frac{e^x}{2}(\sin x-\cos x)+C.$$

R% ª0h0W0f0

$$\begin{aligned}\int e^x(\cos x+i\sin x)\,dx&=\int e^{(1+i)x}dx=\frac{1}{1+i}e^{(1+i)x}+C=\frac{1-i}{2}e^x(\cos x+i\sin x)+C\\&=\frac{e^x}{2}(1-i)(\cos x+i\sin x)+C=\frac{e^x}{2}\left[\cos x+\sin x+(-\cos x+\sin x)i\right]+C\end{aligned}$$

0n Zep ÆR 0 S 0c0f0

$$\int e^x\sin x\,dx=\frac{e^x}{2}(\sin x-\cos x)+C'.$$

O<0_0 0F0j0¿0⁄0 0h0W0fk!0n0 0n0 ″ 0g0B0 0

O **3.14**

$$\begin{aligned}I&=\int\sqrt{x^2+k}\,dx=\int(x)'\cdot\sqrt{x^2+k}\,dx=x\sqrt{x^2+k}-\int x\frac{x}{\sqrt{x^2+k}}\,dx\\&=x\sqrt{x^2+k}-\int\frac{x^2+k-k}{\sqrt{x^2+k}}\,dx=x\sqrt{x^2+k}-\int\sqrt{x^2+k}+k\int\frac{1}{\sqrt{x^2+k}}\,dx\end{aligned}$$

0 0

$$I=\frac{1}{2}\left(\sqrt{x^2+k}+k\int\frac{dx}{\sqrt{x^2+k}}\right)+C.$$

} u(0n ^ L0h0W0f

$$I = \int \sqrt{a^2 - x^2}\, dx$$

0LOB0 0

ŁR zMR 0go8S _ 0 _ 0 0h0D0FUO L0 0B0 0

O **3.15**

$$I_n = \int_0^{\pi/2} \sin^n \theta d\theta \quad (n \in \mathbf{Z},\ n \geq 0)$$

0h0Y0 0h0M0

$$I_n = \int_0^{\pi/2} (-\cos\theta)' \sin^{n-1}\theta\, d\theta = -\left[\cos\theta \sin^{n-1}\theta\right]_{\theta=0}^{\theta=\pi/2} + \int_0^{\pi/2} \cos\theta \cdot (n-1)\sin^{n-2}\theta\cos\theta\, d\theta$$

$$= (n-1)\int_0^{\pi/2} \cos^2\theta \sin^{n-2}\theta\, d\theta = (n-1)\int_0^{\pi/2} (1-\sin^2\theta)\sin^{n-2}\theta\, d\theta = (n-1)(I_{n-2} - I_n)$$

0 0

$$nI_n = (n-1)I_{n-2}.$$

0 0H0k

$$I_n = \frac{n-1}{n} I_{n-2}.$$

N e„

$$I_0 = \int_0^{\pi/2} d\theta = \frac{\pi}{2}, \quad I_1 = \int_0^{\pi/2} \sin\theta\, d\theta = [-\cos\theta]_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = 1$$

0g0B0 0K0 0

$$I_{2k+1} = \frac{2k}{2k+1} I_{2k-1} = \frac{2k}{2k+1}\frac{2k-2}{2k-1} I_{2k-3} = \cdots = \frac{2k\cdot(2k-2)\cdots 2}{(2k+1)\cdot(2k-a)\cdots 3} I_1 = \frac{(2k)!!}{(2k+1)!!},$$

$$I_{2k} = \frac{2k-1}{2k} I_{2k-2} = \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2} I_{2k-4} = \cdots = \frac{(2k-1)(2k-3)\cdots 3\cdot 1}{(2k)(2k-2)\cdots 4\cdot 2} I_0 = \frac{(2k-1)!!}{(2k)!!}\frac{\pi}{2}.$$

^ L0h0W0f(^ 'zMR 0h0j0c0f0W0~0F0L)0

$$I_n = \int_0^{\infty} e^{-x} x^n dx.$$

0~0_

$$I_n = \int \frac{1}{(x^2+a^2)^n} dx$$

0ct N 0 ″ 0g0B0 0 0S0 0k0d0D0f0c0 T} L4.2 0 0 0

4 g t ¢ep0nzMR

4.1 g t ¢ep0nN [zMR 0cR {I ¢ep0g h0 0[0 — **Leibniz** 0n[t

Leibniz (Gottfried Wilhelm Leibniz, 1646–1716) 0o 1702, 1703 ^t0kvz h0W0_2 }Ł0n e 0g0 g t ¢ep0nzMR 0IR {I ¢ep0k0j0 0S0h0 y:0W0_0 0 0c0h0 [O´epY _ 0L1 k!, 2 k!V [P0nzM 0kR ª0U0 0 0h0D0F[t (Nªep[f0nWœg,[t) 0n <f 0c0 0c0h_ (1799, Gauss) 0g0B0 0

[t **4.1** (g t ¢ep0nN [zMR) g t ¢ep0nN [zMR 0cg t ¢ep0h[ep ¢ep0 Arctan 0g h 0 0U0 0 0

4.2 [t 4.1 0n <f

g t _

$$R(x) = \frac{g(x)}{f(x)}$$

0L0B0 0h0M0 0~0ZR [P $g(x)$ 0 R k˝ $f(x)$ 0gRr0c0_UF0hOY0 0 0]0 0^0 $q(x), r(x)$ 0h0Y0 0 0d 0~0

$$g(x) = f(x)q(x) + r(x), \quad \deg r(x) < \deg f(x) \quad 0\sim0_0o \quad r(x) = 0.$$

0Y0 0h

$$R(x) = \frac{q(x)f(x) + r(x)}{f(x)} = q(x) + \frac{r(x)}{f(x)}.$$

0 0H0k

$$\int R(x)\,dx = \int q(x)\,dx + \int \frac{r(x)}{f(x)}\,dx.$$

$q(x)$ 0oY _ 0j0n0g0]0nzMR 0o[„f 0k[L0g0M0 0 0]0S0gN N 0c00 R [P0nk!ep < R k˝0n k!ep0 0h0j0 g t _ $\frac{r(x)}{f(x)}$ 0n { 0 0H0 0

k!0n\ { 0g ‹f 0Y0 ŁR R ep0x0nR ª(ŁR R ep\U) 0 u(0D0 0p0

$$\int \frac{dx}{(x-a)^n} \quad (a\ 0o[\ ep0n[\ ep,\ n\ 0o\ \mathbb{R}q6ep)$$

0h

$$\int \frac{x\ 0n1\ k!_}{p(x)^n}dx \quad (p(x)\ 0o[\ ep0n\{\tilde{}V\ 0g0oV\ epR\ ª0g0M0j0D2\ k!_\ ,\ n\ 0o\ \mathbb{R}q6ep)$$

0 lB0 0 0S0h0k^0w@0g0M0 0 0S0n0F0aRM 0o| !SX0g0

$$\int \frac{dx}{(x-a)^n} = \begin{cases} \dfrac{-1}{n-1}\dfrac{1}{(x-a)^{n-1}} & (n > 1) \\ \log|x-a| & (n = 1). \end{cases}$$

_ 0k0d0D0f0c0

$$p(x) = a\,(x - (\alpha + i\beta))\,(x - (\alpha - i\beta)) \quad (a,\ \alpha,\ \beta\ 0o[\ ep0g0\ a\beta \neq 0,\ i\ 0o\ ZepSXOM)$$

0hf£0Q0 0n0g0

$$p(x) = a\left((x-\alpha)^2 + \beta^2\right)$$

0h0j0 0K0 0 $x - \alpha = u$ 0h nc 0Y0 0S0h0k0 0c0f0

$$\int \frac{u\ 0n1\ k!_}{(u^2+\beta^2)^n}\,du$$

0k^0w@0U0 0 0 0S0nzMR 0ok!0kc†0R0 o8S _ 0 u(0D0f { 0g0M0 0

T} L **4.2** (1) b 0 [ep, n 0 Œq6ep0h0Y0 0h0M0

$$\int \frac{x}{(x^2+b^2)^n}\,dx = \begin{cases} \dfrac{-1}{2(n-1)}\dfrac{1}{(x^2+b^2)^{n-1}} & (n>1) \\ \dfrac{1}{2}\log(x^2+b^2) & (n=1). \end{cases}$$

(2) b 0 0 0g0j0D[ep0h0Y0 0h0M0

$$I_n = \int \frac{1}{(x^2+b^2)^n}\,dx \quad (n \text{ 0o } Œq6ep)$$

0h0J0C0h0

$$I_n = \begin{cases} \dfrac{1}{b^2}\left(\dfrac{x}{(2n-2)(x^2+b^2)^{n-1}} + \dfrac{2n-3}{2n-2}I_{n-1}\right) & (n>1) \\ \dfrac{1}{b}\operatorname{Arctan}\left(\dfrac{x}{b}\right) & (n=1). \end{cases}$$

<f

(1) $u = x^2+b^2$ 0h nc 0Y0 0p| !SX0ky:0[0 0

(2) $n=1$ 0n0h0M0of 0 0K0g0B0 ($x = b\tan\theta$ 0h nc 0[0)0 $n>1$ 0n0h0M0c0 I_{n-1} 0k[0W0f 0 c0CQœ0f0C0 0 ŁR zMR { 0 L0c0f

$$\begin{aligned} I_{n-1} &= \int (x^2+b^2)^{1-n}\,dx = \int (x)'\cdot(x^2+b^2)^{1-n}\,dx \\ &= x(x^2+b^2)^{1-n} - \int x\cdot(1-n)(x^2+b^2)^{-n}(2x)\,dx \\ &= \frac{x}{(x^2+b^2)^{n-1}} + 2(n-1)\int \frac{x^2}{(x^2+b^2)^n}\,dx \\ &= \frac{x}{(x^2+b^2)^{n-1}} + 2(n-1)\int \frac{x^2+b^2-b^2}{(x^2+b^2)^n}\,dx \\ &= \frac{x}{(x^2+b^2)^{n-1}} + 2(n-1)(I_{n-1} - b^2 I_n). \end{aligned}$$

yß 0W0f

$$2(n-1)b^2 I_n = \frac{x}{(x^2+b^2)^{n-1}} + (2n-3)I_{n-1}.$$

0 0H0k

$$I_n = \frac{1}{2(n-1)b^2}\left(\frac{x}{(x^2+b^2)^{n-1}} + (2n-3)I_{n-1}\right). \blacksquare$$

4.3 ŁR R ep0x0nR a

4.3.1 e„

$R(x)$ 0 R [P0nk!ep0lR k˝0nk!ep0 0 \ 0U0Dg t _ (N N 0 O¿[v 0kw R ep_ 0hT|0v0S0h0k 0Y0) 0h0Y0 :

$$R(x) = \frac{g(x)}{f(x)}, \quad \deg g(x) < \deg f(x).$$

ŁR R ep0x0nR ª0n[t 0 y:0W0f0 0]0n <f 0 N 0H0 0S0h0o[0o| !SX0g0B0 0L0 R 0 0f[f0v X4T 0k0o[t 0nt ª0]0n0 0n0L}Pi¸ ª0W0D0n0g0 0S0S0g0o 0 0c0f(lz0W0fg w 0h0D0F0 0Q 0g0c0j0D) ‹f 0Y0 0

4.3.2 R k˝0nV epR ª

R k˝ $f(x)$ 0nV epR ª0LJO L0k0j0 0L0 Y _ 0nV epR ª0k0d0D0f0c0 0~0Zk!0LWœg,v 0g 0B0 0

Nªep[f0nWœg,[t (0 0Ø0 0 0 0°0 0‹0f0„)

} epO´ep0n n k!Y _ $f(x)$ 0o n P 0n } ep0nh90 c 0d0 0Y0j0 0a

$$f(x) = A(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) \quad (A,\ \alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n \text{ 0o } \} \text{ ep})$$

0h1 k!_ 0nzM0h0W0f h0 0[0 0

(\ 0j0C0h0 N 0d0nh9 α_1 0 c 0d0S0h0lR 0K0 0p0 V ep[t 0K0 $f(x) = (x-\alpha_1)f_1(x)$ 0hfŁ0Q0 0S0h0lR 0K0 0 _ 0o^0} l 0g| !SX0g0B0 0 \ 0j0C0h0 N 0d0nh90 c 0d0S0h0n ‹f 0c0]0 0j0 0k ª0W0C0 0S0S0g0ow ue0Y0 0 } Q ep (¢ep 0 } ªg) 0ng,0k 0c0f0D0 0S0h0LY 0D0)

0 h90h0c0 a 0LY _ $f(x)$ 0nh90g0B0 0h0c0 $f(x)$ 0L $x-a$ 0gRr0 R 0 0 0S0h0 0D0F0 a 0L e„z _ $f(x)=0$ 0n ª0g0B0 0h0c0 $f(a)=0$ 0Ib 0 z¸0d0S0h0 0D0F0 V ep[t 0k0 0 0 0S0nN 0d0ngaN 0cT P$0g0B0 0 s W(0n h!ep[f0g0o(Y _ 0n) 0 h9(root)0 0h0D0F I0 0 0Z0 e„z _ 0r0 ª0 0 ‘0Q0gn 0~0[0 0 0W0D0 N! 0cg,_S0cup0j0 i _ı0g0 \ 0j0C0h0 0¸0 0¢0 0„0o U0F0 0n0g00 ˝ ª0 0ol c 0a‘"0D0h0D0Fep[f 0LY 0D0h‘ 0F0

}00K0D0S0h0g0B0 0L0 $f(x)$ 0 1 k!_ 0nzM0kR ª0Y0 N e„0cN 0 0g0B0 0

$f(x)$ 0nh9 a 0L0a0 0F0i m k!0n ˝h90g0B0 0h0c0

$$f(x) = (x-a)^m g(x), \quad g(x) \text{ 0o } x-a \text{ 0gRr0 0j0DY } _$$

0h0j0 0S0h0g0B0 0L0 k!0n0 0F0j\ ¢ep0 u(0D0_R$[l 0L0B0 0

˝h90n ˝ ^f0nR$[

a 0L $f(x)$ 0n (0a0 0F0i) m ˝h9 $\Leftrightarrow$ $f(a) = f'(a) = \cdots = f^{(m-1)}(a) = 0$ 0K0d $f^{(m)}(a) \neq 0$.

0U0f0 $f(x)$ 0L[O´epY _ 0g0B0 0h0M0 $f(x)$ 0n[ep0n{˜V 0g0nV epR ª0 0H0 0F0 [O´epY _ 0k ¢0W0f0ck!0L ˝ 0g0B0 0

[O´epY _ 0nh9

Zep $a = \alpha + i\beta$ ($\alpha, \beta \in \mathbf{R}$) 0L[O´epY _ $f(x)$ 0nh90g0B0 0h0M0 Qq_y } ep $\overline{a} = \alpha - i\beta$ 0 $f(x)$ 0nh90g0B0 0 0U0 0k a 0h $\overline{a}$ 0n ˝ ^f0o{ I0W0D0

($\overline{a+b} = \overline{a} + \overline{b}$, $\overline{a \cdot b} = \overline{a} \cdot \overline{b}$ 0 0 0 $f(x)$ 0L[O´epY _ 0g0B0 0p0 Nßa 0n } ep a 0k[0W0f $\overline{f}(a) = f(\overline{a})$ 0h0j0 0S0h0lR 0K0 0n0g0 a 0L $f(x)$ 0nh90g0B0 0p0 $\overline{a}$ 0 $f(x)$ 0nh90g0B0 0S0h0L R 0K0 0)

[O´epY _ $f(x)$ 0L Zh9 $a = \alpha + i\beta$ 0 c 0a0 0]0n ˝ ^f0 m 0h0Y0 0h0M0 $\overline{a} = \alpha - i\beta$ 0 ˝ ^f m 0nh90g0B0 0K0 0 $f(x)$ 0o

$$(x-a)^m (x-\overline{a})^m = [(x-(\alpha+i\beta))^m]\,[(x-(\alpha-i\beta))^m] = [(x-(\alpha+i\beta))\,(x-(\alpha-i\beta))]^m$$
$$= \{[(x-\alpha)+i\beta]\,[(x-\alpha)-i\beta]\}^m = \left[(x-\alpha)^2+\beta^2\right]^m$$

0gRr0 R 0 0 0S0h0k0j0 0

T} L **4.3** ([O´epY _ 0n[ep0n{˜V 0g0nV epR ª) $f(x)$ 0 n k![O´epY _ 0h0Y0 0 $f(x)$ 0nv łup0j0 h90 $a_1,\ a_2,\ \cdots,\ a_k,\ a_{k+1},\ \overline{a_{k+1}},\ a_{k+2},\ \overline{a_{k+2}},\ \cdots,\ a_\ell,\ \overline{a_\ell}$, 0_0 ‘0W $a_j \in \mathbf{R}$ $(j = 1, 2, \cdots, k)$, $a_j \notin \mathbf{R}$ $(j = k+1, k+2, \cdots, \ell)$ 0h0J0M0 a_j 0n ´ ^f0 m_j, $a_j = \alpha_j + i\beta_j$ $(\alpha_j,\ \beta_j \in \mathbf{R})$ 0hf ł0C0h0M0 k!_ 0Lb 0 z¸0d0

$$(1)\qquad f(x) = a(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_k)^{m_k} \times \left[(x-\alpha_{k+1})^2+\beta_{k+1}^2\right]^{m_{k+1}}\left[(x-\alpha_{k+2})^2+\beta_{k+2}^2\right]^{m_{k+2}}\cdots\left[(x-\alpha_\ell)^2+\beta_\ell^2\right]^{m_\ell}.$$

<f

$$\begin{aligned} f(x) &= a(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_k)^{m_k} \\ &\quad\times (x-(\alpha_{k+1}+i\beta_{k+1}))^{m_{k+1}}(x-(\alpha_{k+1}-i\beta_{k+1}))^{m_{k+1}} \\ &\quad\times (x-(\alpha_{k+2}+i\beta_{k+2}))^{m_{k+2}}(x-(\alpha_{k+2}-i\beta_{k+2}))^{m_{k+2}} \\ &\quad\times\cdots\times (x-(\alpha_\ell+i\beta_\ell))^{m_\ell}(x-(\alpha_\ell-i\beta_\ell))^{m_\ell} \end{aligned}$$

0hf ł0Q0 0S0h0K0 R 0K0 0 ■

S $\prod$ 0 u(0D0 0p(1) 0o

$$\begin{aligned} f(x) &= a\prod_{j=1}^{k}(x-a_j)^{m_j}\prod_{j=k+1}^{\ell}\left[x-(\alpha_j+i\beta_j)\right]^{m_j}\left[x-(\alpha_j-i\beta_j)\right]^{m_j} \\ &= a\prod_{j=1}^{k}(x-a_j)^{m_j}\prod_{j=k+1}^{\ell}\left[(x-\alpha_j)^2+\beta_j^2\right]^{m_j} \end{aligned}$$

0h0ł0 0 0fl0¨0kf ł0Q0 0

4.3.3 N 0D0k} 0jR k˝0 c 0dR ep_ 0x0nR ª

{F 0ck!0Lg 0 ´ 0j u0g0B0 0h 0H0f0D0 0

N 0D0k} 0jR k˝0 c 0dR ep_ 0x0nR ª

g t _ $R(x) = \dfrac{g(x)}{f(x)}$ 0nR k˝ $f(x)$ 0L $f(x) = f_1(x)f_2(x)$ ($f_1(x)$ 0h $f_2(x)$ 0dN 0D0k}) 0hV epR ª0g0M0 0h0M0

$$R(x) = \frac{g(x)}{f_1(x)f_2(x)} = \frac{g_1(x)}{f_1(x)} + \frac{g_2(x)}{f_2(x)} \quad (g_1(x),\ g_2(x)\ 0\text{oY}\ \ _)$$

0hR ª0g0M0 0 $\deg g(x) < \deg f(x)$ 0g0B0 0h0M0 $\deg g_1(x) < \deg f_1(x)$ 0K0d $\deg g_2(x) < \deg f_2(x)$ 0h0j0 0 0F0k $g_1(x),\ g_2(x)$ 0LS 0 0 0 (0_0 ‘0WY _ 0L0 0g0B0 0h0M0 0]0nk!ep 0o $-\infty$ 0h 0H0 0S0h0k0Y0 0)

0S0nT} L0oY 0C0nU0 L0k_ u(0U0 ^^80k ´ 0g0B0 0L0 0S0S0g0o <f 0 w ue0Y0 (hBu0 [3] 0 0)0

0S0 0K0 0 $f(x)$ 0L(1) 0n0 0F0kV epR ª0U0 0 0h0M0

$$f(x) = \sum_{j=1}^{k}\frac{(m_j-1\ \text{k!N N 0nY}\quad_)}{(x-a_j)^{m_j}} + \sum_{j=k+1}^{\ell}\frac{(2m_j-1\ \text{k!N N 0nY}\quad_)}{\left[(x-\alpha_j)^2+\beta_j^2\right])^{m_j}}$$

0hR ª0U0 0 0S0h0IR 0K0 0

4.3.4 Y _ 0n $(x-a)$ 0n0 0 x0x0n\U

$$R(x)=\frac{P(x)}{(x-a)^m},\quad P(x)\ 0o\ m-1\ k!N\ N\ 0nY\quad _$$

0k[0W0f0 R [P $P(x)$ 0 $(x-a)$ 0n0 0 x0k\U 0Y0 0 0d0~0

(2) $P(x)=c_1(x-a)^{m-1}+c_2(x-a)^{m-2}+\cdots+c_{m-1}(x-a)+c_m$ $(c_1,\cdots,c_m$ 0o[ep)

0h0g0M0 0p0

$$\begin{aligned}R(x)&=\frac{c_1(x-a)^{m-1}+c_2(x-a)^{m-2}+\cdots+c_{m-1}(x-a)+c_m}{(x-a)^m}\\&=\frac{c_1}{x-a}+\frac{c_2}{(x-a)^2}+\cdots+\frac{c_{m-1}}{(x-a)^m}\end{aligned}$$

0hR a0g0M0 0
$\{c_j\}_{j=1}^m$ 0nlB0 e„0k0cN 0 0B0 0

(1) (k!ep0nON0D 0nO´ep0K0 lB0 0) $Q_m(x)=P(x)$ 0h0J0M0 $i=m,\ m-1,\cdots,1$ 0n 0k $Q_i(x)$ 0 $x-a$ 0gRr0c0_UF0 $Q_{i-1}(x)$, OY0 0 c_i 0h0J0C0h0 $\{c_j\}_{j=1}^m$ 0LlB0~0 0

(2) (k!ep0n 0D 0nO´ep0K0 lB0 0) $P_1(x)=P(x)$ 0h0J0M0 $i=1,\cdots,m$ 0n 0k$P_i(x)$ 0ng k!0nO´ep0 c_i 0h0W0 $P_i(x)-c_i(x-a)^{m-i}$ 0 $P_{i+1}(x)$ 0h0Y0 0

4.3.5 Y _ 0n 2 k!_ 0n0 0 0x0 0x0n\U

$$R(x)=\frac{P(x)}{p(x)^m},\quad p(x)=(x-\alpha)^2+\beta^2,\quad P(x)\ 0o\ 2m-1\ k!N\ N\ 0nY\quad _$$

0L0B0 0h0W0 0F0 1 k!_ $(x-a)$ 0n0 0 x0x0n\U 0nw O<0 0W0f0 $Q_m(x)=P(x)$ 0h0J0M0 $i=m,$ $m-1,\cdots,1$ 0n 0k$Q_i(x)$ 0 $p(x)$ 0gRr0c0_UF0 $Q_{i-1}(x)$, OY0 0 (1 k!N N 0n_ 0j0n0g) c_ix+d_i 0h0J0S0F0

$$\begin{aligned}R(x)=Q_m(x)&=Q_{m-1}(x)p(x)+(c_mx+d_m),\\Q_{m-1}(x)&=Q_{m-2}(x)p(x)+(c_{m-1}x+d_{m-2}),\\&\vdots\\Q_i(x)&=Q_{i-1}(x)p(x)+(c_ix+d_i),\\&\vdots\\Q_2(x)&=Q_1(a)p(x)+(c_2x+d_2),\\Q_1(x)&=c_1x+d_1\end{aligned}$$

0h0j0 ($Q_i(x)$ 0o $2i-1$ k!N N 0n_ 0j0n0g0 $Q_1(x)$ 0~0g0gS´_'0ckb0)0

$$P(x)=(c_1x+d_1)p(x)^{m-1}+(c_2x+d_2)p(x)^{m-2}+\cdots+(c_{m-1}x+d_{m-1})p(x)+(c_mx+d_m)$$

0h0j0 0K0 0

(3) $$R(x)=\frac{c_1x+d_1}{p(x)}+\frac{c_2x+d_2}{p(x)^2}+\cdots+\frac{c_mx+d_m}{p(x)^m}$$

4.3.6 0~0h0

T} L **4.4** $f(x)$ 0L(1) 0n0 0F0kV epR ª0U0 0 0h0M0 $R(x)$ 0ck!0n0 0F0kf‡0Q0 0

$$R(x) = \sum_{j=1}^{k} \sum_{m=1}^{m_j} \frac{b_{j,m}}{(x-a_j)^m} + \sum_{j=k+1}^{\ell} \sum_{m=1}^{m_j} \frac{c_{j,m}x + d_{j,m}}{\left[(x-\alpha_j)^2 + \beta_j^2\right]^m} \quad (b_{j,m},\ c_{j,m},\ d_{j,m}\ \text{0o i_S0j[ep}).$$

0S0nT} L0nv c¥v 0ßR {Iv 0j <f 0LgImf[1] 0k 0c0f0D0 0

4.4 [t **4.1** 0n|ß

r0 0jzMR 0Lg t ¢ep0nzMR 0k^0w@0g0M0 0n0g0 R {I ¢ep0gzMR S 0g0B0 0S0h0IR 0K0 0 N N $R(x)$ 0 $R(x,y)$ 0cg t _ 0h0Y0 0

4.4.1 $R(e^x)$ 0nzMR

$$I = \int R(e^x)\,dx$$

0k[0W0f0 $u = e^x$ 0h0J0C0h0 $du = e^x dx$ 0 0 $dx = du/u$ 0 ‘0K0

$$I = \int \frac{R(u)}{u}\,du.$$

0S0 0cg t ¢ep0nzMR 0g0B0 0K0 0 R {I ¢ep0g h0 0[0 0

4.4.2 $R(x, \sqrt{ax+b})$ 0nzMR

$$I = \int R\left(x, \sqrt{ax+b}\right)\,dx$$

0k[0W0f0 $u = \sqrt{ax+b}$ 0h0J0C0h0 $u^2 = ax+b$ 0g0B0 0K0 0

$$x = \frac{u^2-b}{a}, \quad dx = 2udu$$

0h0j0 0

$$I = 2\int R\left(\frac{u^2-b}{a}, u\right) u\,du.$$

0S0 0 g t ¢ep0nzMR 0g0B0 0K0 0 R {I ¢ep0g h0 0[0 0

4.4.3 N ¢ep0ng t ¢ep

0U0f0

$$t = \tan\frac{x}{2}$$

0h0J0C0h0 P 0nQ1_ 0K0

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2},$$
$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}.$$

0~0_

$$dt = \frac{1}{2\cos^2\frac{x}{2}} = \frac{1}{2}\left(\tan^2\frac{x}{2} + 1\right)dx$$

0 0

$$dx = \frac{2}{1+t^2}dt.$$

(0B0 0D0o $x = 2\operatorname{Arctan} t$ 0n_fiR 0K0 lB0 0 0S0h0 0g0M0 0)

| ß **4.5** (N ¢ep0ng t _ 0nzMR) $R(X,Y)$ 0L X, Y 0ng t _ 0g0B0 0h0M0

$$\int R(\cos x, \sin x)\,dx = \int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)\frac{2}{1+t^2}\,dt.$$

0 0H0kN ¢ep0ng t _ $R(\cos x, \sin x)$ 0nN [zMR 0cg t ¢ep0nN [zMR 0k^0w@0U0 0 $t = \tan(x/2)$ 0ng t ¢ep0 [ep ¢ep0 Arctan 0g h0 0U0 0 0

4.4.4 2 k!q!t ¢ep

| ß **4.6** (**2** k!q!t ¢ep) $R(X,Y)$ 0L[g t _ 0g0B0 0h0M0

$$\int R\left(x, \sqrt{ax^2+bx+c}\right)dx$$

0cg t ¢ep0nzMR 0k^0w@0g0M0 0

<f ax^2+bx+c 0 ^se„[b 0W0f0K0 Y ep0n1 k!Y c 0 L0F0S0h0g0 $\sqrt{ax^2+bx+c}$ 0ck!0n 3 0d0n0F0a0n0D0Z0 0KN 0d0k0j0 0

(1) $r = \sqrt{a^2x - x^2}$. $x = a\sin\theta$ $(-\pi/2 < x < \pi/2)$ 0h0J0C0 $r = \sqrt{a^2(1-\sin^2\theta)} = a\cos\theta$, $dx = a\cos\theta\,d\theta$ 0h0j0 0

(2) $r = \sqrt{x^2 - a^2}$. $x \geq a$ 0nX4T 0o $x = a/\cos\theta$ $(0 \leq \theta < \pi/2)$ 0h0J0C0 $r = \sqrt{a^2\tan^2\theta} = a\tan\theta$, $dx = \dfrac{a\sin\theta}{\cos^2\theta}d\theta$ 0h0j0 0

(3) $r = \sqrt{x^2 + a^2}$. $x = a\tan\theta$ $(-\pi/2 < x < \pi/2)$ 0h0J0C0 $r = \sqrt{a^2(\tan^2\theta+1)} = \sqrt{a^2/\cos^2\theta} = a/\cos\theta$, $dx = \dfrac{a\,d\theta}{\cos^2\theta}$.

0S0 0K0 N ¢ep0ng t _ 0nN [zMR 0k^0w@0g0M0 0 | ß 4.5 0K0 g t ¢ep0nN [zMR 0k^0w@ 0g0M0 0 ■

4.5 QwOSO

O **4.7**

$$I = \int \frac{dx}{x^2 - 1}.$$

ŁR R ep0x0nR ª0o

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

0h0j0 0

$$I = \frac{1}{2}\int \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx = \frac{1}{2}\left(\log|x-1| - \log|x+1|\right) = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right|. \blacksquare$$

O **4.8**

$$I = \int \frac{x^4}{x^3 - 3x + 2}\, dx.$$

R [P0 R k˝0gRr0c0f0

$$x^4 = x(x^3 - 3x + 2) + 3x^2 - 2x$$

0L_ 0 0 0 0K0 0

$$I = \int \frac{x(x^3-3x+2) + 3x^2 - 2x}{(x^3 - 3x + 2)}\, dx = \int \left(x + \frac{3x^2 - 2x}{x^3 - 3x + 2}\right) dx = \frac{x^2}{2} + J,$$
$$J := \int \frac{3x^2 - 2x}{x^3 - 3x + 2}\, dx.$$

R k˝0o $x^3 - 3x + 2 = (x-1)^2(x+2)$ 0hV epR ª0U0 0 0K0 0

$$\frac{3x^2 - 2x}{x^3 - 3x + 2} = \frac{A}{x+2} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$$

0h0J0Q0 0 R k˝0 bU0c0f0

$$3x^2 - 2x = A(x-1)^2 + B(x+2) + C(x-1)(x+2).$$

$x = 1$ 0 NªQe0W0f

$$1 = 3B \quad \therefore B = \frac{1}{3}.$$

$x = -2$ 0 NªQe0W0f

$$16 = 9A \quad \therefore A = \frac{16}{9}.$$

x^2 0nO´ep0 k 0W0f

$$3 = A + C \quad \therefore C = 3 - A = 3 - \frac{16}{9} = \frac{11}{9}.$$

N N 0 0

$$\frac{3x^2 - 2x}{x^3 - 3x + 2} = \frac{16}{9(x+2)} + \frac{1}{3(x-1)^2} + \frac{11}{9(x-1)}.$$

0 0H0k

$$J = \frac{16}{9}\log|x+2| - \frac{1}{3}\frac{1}{x-1} + \frac{11}{9}\log|x-1|.$$

$$I = \frac{x^2}{2} + \frac{16}{9}\log|x+2| - \frac{1}{3(x-1)} + \frac{11}{9}\log|x-1|. \blacksquare$$

O **4.9** (R k˝0ke } 2 k!_ 0L0B0 X4T)

$$I = \int \frac{x^4 + 2x^3 + 3x^2 + 4x + 5}{x^3 - 1}\, dx.$$

0~0ZR [P0 R k˝0gRr0c0f0

$$x^4 + 2x^3 + 3x^2 + 4x + 5 = (x+2)(x^3 - 1) + 3x^2 + 5x + 7$$

0L_ 0 0 0 0K0 0

$$I = \int \frac{(x+2)(x^3-1) + 3x^2 + 5x + 7}{x^3 - 1}\, dx = \int (x+2)dx + \frac{3x^2 + 5x + 7}{x^3 - 1}\, dx = \frac{x^2}{2} + 2x + J,$$
$$J := \int \frac{3x^2 + 5x + 7}{x^3 - 1}\, dx.$$

$x^3 - 1$ 0n[ep0n{ ˜V 0g0nV epR ª0o

$$x^3 - 1 = (x-1)(x^2 + x + 1)$$

0g0B0 0K0 0

$$\frac{3x^2 + 5x + 7}{x^3 - 1} = \frac{A}{x-1} + \frac{Bx + c}{x^2 + x + 1}$$

0hR ª0g0M0 0c0Z0 R k˝0 bU0c0f

$$3x^2 + 5x + 7 = A(x^2 + x + 1) + (Bx + c)(x - 1). \tag{4}$$

$x = 1$ 0 NªQe0W0f

$$15 = 3A \quad \therefore \quad A = 5.$$

N e„ (4) 0nN! ”0n x^2 0nO´ep0 k 0W0f

$$3 = A + B \quad \therefore B = 3 - A = 3 - 5 = -2.$$

T i 0k (4) 0nN! ”0n[ep 0 k 0W0f

$$7 = A - C \quad \therefore C = A - 7 = 5 - 7 = -2.$$

0 0H0k

$$J = \int \left(\frac{5}{x-1} - \frac{2x+2}{x^2 + x + 1} \right) dx = 5 \log|x - 1| - K, \quad K := \int \frac{2x+2}{x^2 + x + 1} dx.$$

$$K = \int \frac{2x + 1 + 1}{x^2 + x + 1}\, dx = \int \frac{(x^2 + x + 1)'}{x^2 + x + 1}\, dx + \int \frac{1}{x^2 + x + 1}\, dx$$
$$= \log(x^2 + x + 1) + L, \quad L := \int \frac{1}{x^2 + x + 1}\, dx.$$

0U0f0

$$x^2 + x + 1 = \left(x + \frac{1}{2} \right)^2 + \frac{3}{4}$$

0g0B0 0K0 0

$$x + \frac{1}{2} = \frac{\sqrt{3}}{2} \tan\theta$$

0h0J0C0h0

$$x^2+x+1=\frac{3}{4}(1+\tan^2\theta)=\frac{3}{4\cos^2\theta}.$$

$$dx=\frac{\sqrt{3}}{2\cos^2\theta}\,d\theta,\quad \theta=\operatorname{Arctan}\frac{2\left(x-\dfrac{1}{2}\right)}{\sqrt{3}}$$

0h0j0 0

$$L=\int\frac{4\cos^2\theta}{3}\cdot\frac{\sqrt{3}}{2\cos^2\theta}d\theta=\int\frac{2\sqrt{3}}{3}d\theta=\frac{2\sqrt{3}}{3}\theta=\frac{2\sqrt{3}}{3}\operatorname{Arctan}\frac{2\left(x-\dfrac{1}{2}\right)}{\sqrt{3}}.$$

0 0H0k

$$I=\frac{x^2}{2}+2x+5\log|x-1|-\log(x^2+x+1)-\frac{2\sqrt{3}}{3}\operatorname{Arctan}\frac{2\left(x-\dfrac{1}{2}\right)}{\sqrt{3}}.\blacksquare$$

4.6 OY · — [0nzMR {

N N 0n p 0c0 [0cS t v 0j0 0n0g0B0 0 _ u(N s 0 0 0 UO L0k0cN SAR 0g0B0 0
O 0H0pN 0n p 0g0c0 g t ¢ep0nR k˝0ks 0 0 0 Y _ 0nh90LlB0~0c0_0hN [0W0f0D0 0 x" 0K0kY _ 0nh90n[XW(0cN ep[f0nWœg,[t 0gO <0U0 0 0S0h0g0B0 0L0 0]0nQwOSv 0j { l 0o R%0nUO L0g0B0 0

2 k!e„z _ 0nh90nQl_

$$ax^2+bx+c=0\quad(a\neq0)$$

0nh90o

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

0n0 0F0jQl_ 0L_ 0 0 0 0n0c0 e„z _ 0nk!ep0L4 0~0g0g0 5 k!N N 0nN epe„z _ 0k[0W0f0c0 g PV 0nV RG0h] h90 u(0D0_h90nQl_ 0o[XW(0W0j0D0S0h0L <f 0U0 0f0D0 (Ábel, Galois)[3]0 0] 0S0g[0nV epR a0c0 (0D0C0 0g0 | ^f0kQœge0 0h0o 0H) O<v 0k0W0K_ 0 0 0j0D0S0h 0LY 0D0

0 0W[zMR 0nP$0L O<v 0kw 0 0_0D0 'Q0j0 0p0 N [zMR 0 N N¸0[0Z0kv c¥epP$zMR 0 L0F 0n0LO¿R)0j0S0h0LY 0D0

0U0 0kN [zMR 0 { 0Y0 0k0W0f0 0 g 0n0ł0 0 0 0 0¿0 0k0 0 ep_ Q t 0g0c0 N 0k <f 0W0_0 0F0j ŁR R ep0x0nR a0 R)u(0Y0 0n0h0oup0j0 0¢0°0·0Œ0"0 0Lc¡u(0U0 0f0D0 (N°0n0h 0S0 0 R k˝0nq!^se„R a0 R)u(0Y0 **Risch** 0n{ l 0Lg R 0g0B0 0h0D0F)0

5 N [zMR 0L0 lB0~0 0j0D0 X4T

5.1 0 lB0~0 0j0D0 0na Ts

N–[fh!0K0 {I[fh!0 0U0 0kY'[f0nWœy ep[f0~0g0k r0 0j ¢ep0 [f0 0g0M0_0L0 0 0a0 0 0S 0 0g0 0Y0y0f0n ¢ep0 0L h0 0U0 0 0 0Q0g0o0j0D0 0Dc 0H0 0h0S0 0~0g[f0 0 ' ¢ep0 'Q0 O 0c0f h0 0Y0S0h0n0g0M0j0D ¢ep0L0B0 0

[3] Nepe[f0n Galois t 0g[f0v0

Y'0V0c0q0k 0c0f[4]0 0S0 0~0g[f0 0' ¢ep0'0Q0 O 0c0f h0 0Y0S0h0n0g0M0 ¢ep0 R {I ¢ep0h 0 0v0

g t ¢ep0nN [zMR 0cR {I ¢ep0k0j0 0S0h0 [f0 0'0L0 R {I ¢ep0nN [zMR 0gR {I ¢ep0k0o 0j0 0j0D0 0n0L0B0 0n0c0]0 0{0iN ŒRq60k0ca 0X0 0 0j0D0g0B0 0F0

R {I ¢ep0nQhOS0o_fiR 0k0d0D0f0o 0X0f0D0 0L0 zMR 0k0d0D0f0o 0X0f0D0j0D

5.2 iUQ zMR

$R(x, \sqrt{ax+b})$, $R(x, \sqrt{ax^2+bx+c})$ 0LzMR 0g0M0 0S0h0cR 0K0c0_0K0 0 k!0o

$$\int R\left(x, \sqrt{ax^3+bx^2+cx+d}\right) dx, \quad \int R\left(x, \sqrt{ax^4+bx^3+cx^2+dx+e}\right) dx,$$

0 0H0_0C0j0 0L0 [0c0S0n_b0n ¢ep0nN [zMR 0c0 ryk 0jO Y 0 d0D0fR {I ¢ep0g0o h0 0U 0 0j0D0S0h0L <f 0U0 0f0D0 0 0S0 0 0nzMR 0 iUQ zMR 0hT|0v0

0Y0y0f0niUQ zMR 0c0 i_S0jY epY c 0gR {I ¢ep0h0 k!0n3 0d0nzMR

$$F(k,\varphi) = \int_0^{\varphi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}\, dt = \int_0^{\sin\varphi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad (0<k<1),$$

$$E(k,\varphi) = \int_0^{\varphi} \sqrt{1-k^2\sin^2 t}\, dt = \int_0^{\sin\varphi} \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt \quad (0<k<1)$$

$$\Pi(k,n,\varphi) = \int_0^{\varphi} \frac{dt}{(1+n\sin^2 t)\sqrt{1-k^2\sin^2 t}} = \int_0^{\sin\varphi} \frac{dt}{(1+nt^2)\sqrt{(1-t^2)(1-k^2t^2)}}$$

0 u(0D0f h0 0[0 (**Legendre-Jacobi** 0nj n _b)0 0]0 0^0 {,N z.N [QhiUQ zMR 0 {,N z.N [QhiUQ zMR 0 {,N z.N [QhiUQ zMR 0hT|0v0 0S0 0 0cR {I ¢ep0g0o h0 0U0 0j0D0

iUQ zMR 0c0]0n ¢ep(iUQ ¢ep) 0 0H0 0S0h0k0 0 g, ŒOLf 0 0K0k0j0 0S0h0Lw 0 0 0f 0D0 0

$$u = \int_0^{z} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad \Leftrightarrow \quad z = \operatorname{sn}(u,k) = \operatorname{sn} u,$$

$$u = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(k,\varphi) \quad \Leftrightarrow \quad \varphi = \operatorname{am}(u,k) = \operatorname{am} u.$$

iUQ ¢ep0oY ep0 } ep0n{˜V 0k ŒRq60k^ 0R0 0S0h0L0g0M0 N ˝Thg 0 c 0d0S0h0Lry_·0g 0B0 0

0S0n0B0_0 0n0S0h0c0

$$u = \int_0^{z} \frac{dx}{\sqrt{1-x^2}} = \operatorname{Arcsin} z \quad \Leftrightarrow \quad z = \sin u$$

0g0B0 0 sin 0IN ˝Thg 0 c 0d ¢ep0g0B0 0S0h0h c0D[qg0 0j0W0f0D0 0

iUQ ¢ep 0o19 N } ep[f0n 0g0B0 0 C. F. Gauss (0 0⁄0˜, 1777–1855), N. H. Abel (0 ¸0° 0f0S0 , 1802–1829), C. G. J. Jacobi (0 0⁄0˜, 1804–1851) 0n3 N"0Lvz 0h 0 0 0f0D0 0

O **5.1** SXc/[P0nc/0 0n 0 θ 0h0W0 0]0ng Y'P$0 α, Qh0¤0˝0°0fi0 0 E 0h0Y0 0h0

$$\frac{1}{2}m\left(\ell\frac{d\theta}{dt}\right)^2 + mg\ell(1-\cos\theta) = E, \quad E = mg\ell(1-\cos\alpha).$$

[4]R {I ¢ep0 | [˜0k['0Y0 0n0o bP 0g0B0 0 0]0 0 B L0Y0 0S0h0k0]0 0{0ia Ts0L0j0D0h' 0 0 0 0n0g0 0S 0S0g0ow ue0Y0 0

0S0 0K0

$$k = \sin\frac{\alpha}{2}, \quad \sin\frac{\theta}{2} = k\sin\varphi$$

0h0J0C0h0

$$t = \sqrt{\frac{\ell}{g}}F(k,\varphi).$$

ry0kThg T 0o

$$T = 4\sqrt{\frac{\ell}{g}}K(k), \quad K(k) := K\left(k, \frac{\pi}{2}\right).$$

0U0 0k

$$\sin\frac{\theta}{2} = k\,\mathrm{sn}\left(\sqrt{\frac{g}{\ell}}t, k\right). \blacksquare$$

5.3 zMR [ep, zMR kc_&

_ u(N k!0nzMR 0L c0Cs 0 0 0 0 0S0 0 0 R {I ¢ep0g0o h0 0Y0S0h0L0g0M0j0D0

$$\mathrm{Li}(x) = \int_0^x \frac{dt}{\log t}, \quad \mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt.$$

5.4 kc R ^ 0n[˘^ƒ ¢ep0nN [zMR

j n [5]kc R ^ (Gauss R ^) 0n[˘^ƒ ¢ep0o

$$f(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

0g0B0 0 R ^ ¢ep0o

$$\phi(x) = \int_{-\infty}^{x} f(t)\,dt$$

0h0j0 0L0 0S0 0cR {I ¢ep0g0o h0 0U0 0j0D0S0h0IR 0K0c0f0D0 0
0ł0 0 0 0 0¿0 0n OS 0nN z.0g0B0 UNIX 0nep[f ¢ep0Ø0⁄0 0Ø0Œ0f0k0c0 ⁄] ¢ep

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

0 { 0Y0 ¢ep0L˝+0~0 0f0D0 0L0

$$\phi(x) = \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$$

0h0D0F ¢O´0L0B0 0n0g0 ϕ 0n { 0o|!SX0g0B0 0

[5]^sWG0, R ec1 0nkc R ^ 0 j n kc R ^ 0hT|0v0

A S f } ¢ep0n‘’ Œ

$$\cosh^2 x - \sinh^2 x = 1,$$

$$1 - \tanh^2 x = \frac{1}{\cosh^2 x},$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y,$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y,$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y},$$

$$\sinh 2x = 2 \sinh x \cosh y,$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1,$$

$$\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x},$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2},$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2},$$

$$\sinh x + \sinh y = 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2},$$

$$\sinh x - \sinh y = 2 \cosh \frac{x+y}{2} \sinh \frac{x-y}{2},$$

$$\cosh x + \cosh y = 2 \cosh \frac{x+y}{2} \cosh \frac{x-y}{2},$$

$$\cosh x - \cosh y = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2},$$

B S f } ¢ep

$$\cosh^{-1} x = \log(x + \sqrt{x^2 - 1}) \quad (x \geq 1),$$

$$\sinh^{-1} x = \log(x + \sqrt{x^2 + 1}) \quad (x \in \mathbf{R}),$$

$$\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x} \quad (|x| < 1),$$

$$\coth^{-1} x = \frac{1}{2} \log \frac{x+1}{x-1} \quad (|x| > 1),$$

o **B.1** $\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$ 0 y:0 [0

(ª) $y = \sinh^{-1} x$ 0h0J0C0h $x = \sinh y = \dfrac{e^y - e^{-y}}{2}$. $Y = e^y$ 0h0J0C0h $x = (Y - 1/Y)/2$ 0 0

$$Y^2 - 2xY - 1 = 0.$$

0S0n2k!e„z _ 0nh90o $x \pm \sqrt{x^2+1}$ 0g0B0 0L0 $Y > 0$ 0kłła 0Y0 0h $Y = x + \sqrt{x^2+1}$. 0 0H0k

$$\sinh^{-1} x = y = \log Y = \log\left(x + \sqrt{x^2+1}\right). \blacksquare$$

o **B.2** $\cosh^{-1} x = \log\left(x + \sqrt{x^2-1}\right)$ $(x \geq 1)$ 0 y:0[0

(ª) $\cosh\colon \mathbf{R} \to \mathbf{R}$ 0cQhSX\ 0g0j0D0 R6 P0W0_ $f\colon [0,\infty) \ni x \mapsto \cosh x \in [1,\infty)$ 0cQhSX\ 0k0j 0 0 0S0n ¢ep0L $\cosh^{-1}$ 0g0B0 0

$y = \cosh^{-1} x$ 0h0J0C0h $x = \cosh y = \dfrac{e^y + e^{-y}}{2}$. N 0g 0y0_0 0F0k $y \geq 0$ 0h0Y0 0n0L} g_0g 0B0 0 0~0_ $x \geq 1$ 0h0j0 0S0h0kłła 0Y0 0

$Y = e^y$ 0h0J0C0h0 $Y \geq 1$, $x = (Y + 1/Y)/2$. _ 0 0

$$Y^2 - 2xY + 1 = 0.$$

0S0n2k!e„z _ 0o2[h9 $Y_1 = x + \sqrt{x^2-1}$, $Y_2 = x - \sqrt{x^2-1}$ 0 c 0d0L0 [0o $x > 1$ 0k[0W0f $Y_1 > 1$ 0K0d $Y_2 < 1$ 0g0B0 0n0g(h90hO´ep0n ¢O´0K0 $Y_1 Y_2 = 1$ 0g0B0 0 f 0 0K0k $Y_1 > 1$ 0j0n 0g $Y_2 < 1$) Y_2 0cN i0g0 $Y = Y_1 = x + \sqrt{x^2-1}$. 0 0H0k

$$\cosh^{-1} x = y = \log Y = \log\left(x + \sqrt{x^2-1}\right). \blacksquare$$

UO tanh 0n ¢ep0g0B0 $\tanh^{-1}$ 0k0d0D0fN N 0nUC0k{T0H0

(1) $\tanh^{-1} x = \dfrac{1}{2}\log\dfrac{1+x}{1-x}$ 0 y:0[(2) $\left(\tanh^{-1} x\right)' = \dfrac{1}{1-x^2}$ 0 y:0[

S´ e s.

[1] gImfQIY+, ªg Qe I, II, goN<Y'[fQœrHO (1980, 198?).

[2] b8u0 v T , iUQ ¢epQe , e g, U y>(2001).

[3] hBu0 yPS , Y _ 0¸0 0¨, http://nalab.mind.meiji.ac.jp/~mk/note/polynomial.pdf